**Equation of Motion**

**Question-01:** Derive the Euler’s equation of motion.

**OR**

Established the equation of motion and derive Lamb’s hydrodynamical equation.

**Answer:** Consider a closed surface in the moving fluid such that it encloses a volume. Within this surface consider any point  and let be the density of the fluid particle at  and  be the elementary volume enclosing .











The mass  of the element at  always remains constant. If  be the velocity at then the momentum  of the volume is,



The rate of change of momentum is,







Again let be the external force per nit mass acting on the fluid. The total force on volume is,



If  be the pressure along the outward drawn unit normal  of the element then the total surface force is,





Now Newton’s second law,







This is true for all volume if





This is Euler’s equation of motion.

Since  so equation (4) can be written as,



Also we have



 

The equation (5) reduces to,



 



This is called Lamb’s hydrodynamical equation.

**Conservative Field of Force:** In a conservative field, the work-done by the force of the field in taking a unit mass from  to is independent of the path.

F

E

D

C

B

A

Thus we have,



where  is a scalar point function and is known as potential function.

**Question-02:** Derive pressure equation for irrotational motion of a fluid.

**OR**

Derive Bernoulli’s equation in its most general form.

**OR**

Derive Bernoulli’s equation for irrotational motion of an incompressible fluid.

**OR**

Discuss the different aspects of motion under conservative body force.

**Answer:** The Euler’s equation of motion is,



Since  so equation (1) can be written as,



Also we know,





The equation (2) reduces to,



Let us consider the motion is irrotational and the body forces are conservative.

So, .

Putting these in (3) we get,





If the density is a function of pressure only i.e. , then we consider following relation,



Now, 











Using this value in (4) we get,





which is true if,



where denotes an instantaneous constant, i.e. a function of  only and has the same value throughout the fluid.

This equation is called the pressure equation for irrotational motion of a fluid. This is also called Bernoulli’s equation in its most general form.

If the density is constant, i.e.  then the equation (5) reduces to,



This is called Bernoulli’s equation for the unsteady irrotational motion of an incompressible fluid.

If the motion is steady i.e. , then equation (6) becomes,



This is called Bernoulli’s equation for the steady irrotational motion of an incompressible fluid.

**Question-03:** State and prove Bernoulli’s theorem for a compressible fluid.

**Statement:** This theorem states that, if the motion of a compressible fluid is steady and the velocity potential does not exists, then



where  is the potential function from which the external forces are derivable.

**Proof:** The Euler’s equation of motion is,



Since  so equation (1) can be written as,



Also we know,





The equation (2) reduces to,



For steady motion  so the equation (3) reduces to,



If is derivable from some potential function say,  then we have,



Putting this in (4) we get,











whence

. **(Proved)**

**Problem**

**Problem-01:** Air obeying Boyle’s law is in motion in a uniform tube of small section. Prove that if  be the density and  the velocity at a distance  from a fixed point at time 

.

**Solution:** Let  be the density and  be the velocity at a distance  from the end of the tube at any time . The equation of motion and the equation of continuity is given by



and 

Since the air obeys Boyle’s law, then



[Boyle’s law: At constant temperature the pressure is inversely proportional to the volume or proportional to the density]

From (1) and (3), we have



Differentiating (2) partially with respect to , we have







From (2), (4) and (5), we have







 **(Proved)**

**Problem-02:** An elastic fluid, the weight of which is neglected, obeying Boyle’s law is in motion in a uniform straight tube. Prove that on the hypothesis of parallel sections the velocity at any time  at a distance  from a fixed point in the tube is defined by the equation

.

**Solution:** Let  be the density and  be the velocity at a distance  from a fixed point in the tube at any time . The equation of motion and the equation of continuity is given by



and 

Since the fluid obeys Boyle’s law, then



[Boyle’s law: At constant temperature the pressure is inversely proportional to the volume or proportional to the density]

From (1) and (3), we have



Differentiating (4) partially with respect to , we have















 **(Proved)**

**Problem-03:** A pulse travelling along a fine straight uniform tube filled with gas causes the density at any time  and distance  from the origin where the velocity is  to become . Prove that the velocity (at time  and distance  from the origin) is given by

.

**Solution:** Let  be the density of the gas at a distance, and be the velocity there, then we have



The equation of continuity is given by





Now by the given condition,



and 

Substituting these values (1), we get







Integrating,





where is an integrating constant.

Initially when  then 

Applying this condition in (2), we get



From (2), we have





 **(Proved)**

**Problem-04:** A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are and ; if and  be the corresponding velocities of the stream and if the motion be supposed to be that of the divergence from the vertex of the come, prove that



where  is the pressure divided by the density and supposed constant.

**Answer:** Let and  be the ends of the conical pipe such that  and . Also let  and be the densities of the stream at the ends and  respectively.





















Hence the equation of continuity is





By Bernoulli’s theorem (in absence of external forces like gravity), we have



But .

The equation (2) becomes,



Integrating, 

when ,  then equation (4) reduces to,



when ,  then equation (4) reduces to,



From(6) and (5), we get











From equation (1), we get

 (**Proved**)

**Problem-05:** A stream in a horizontal pipe, after passing a contraction in the pipe at which its area is , is delivered at atmospheric pressure at a place where the sectional area is . Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth below the pipe; being the delivery per second.

**Answer:** Let  and be the velocity and pressure at . Also let and be the velocity and pressure at .











Hence the equation of continuity is





By Bernoulli’s theorem (in absence of external forces like gravity) for incompressible fluid, namely



we obtain









Let be the height through water is sucked up. If be the cross section of the tube then





From (2) and (3), we have



 (**Showed**)

**Problem-06:** A quantity of liquid occupies a length of a straight tube of uniform bore under the action of force which is equal to  to a point in the tube, where  is the distance from . Find the motion and show that if be the distance of the nearer free surface from , pressure at any point is given by



**OR**

A quantity of liquid occupies a length of a straight tube of uniform small bore under the action of a force to a point in the tube varying as a distance from that point. Determine the pressure at any point.

**Solution:** Let  be the pressure and the velocity at a distance  from the fixed point ; and let be the distance of the nearer surface from . Then the equation of continuity is



Let  be the external force at a distance  which acts towards . Then the equation of motion



gives 

Integrating (2) with respect to , we get



But  when  and . So (3) gives



and 

Subtracting (4) from (5), we get







Putting , so that , (7) gives



Whose solution is 

Since , it yields



in which A and B may be determined from the knowledge of initial position and velocity.

We now determine pressure from (4), we get



Putting this value of C in (3), we get



Using (6) in (9), we get



which gives the pressure at any point.